

## On QCD resummation with $k_t$ clustering

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**ABSTRACT:** We revisit the impact of the jet algorithm on predictions of energy flow into gaps between hard jets, defined using the  $k_t$  clustering procedure. The resulting prediction has two distinct components: a primary emission piece that is related to independent emission of soft gluons by the hard jets and a correlated emission (non-global) piece known only in the large  $N_c$  limit. We analytically compute the dependence of the primary emission term on the jet algorithm, which gives significantly more insight than our previous numerical study of the same. We also point out that the non-global component of the answer is reduced even more significantly by the clustering than suggested previously in the literature. Lastly we provide improved predictions for the latest ZEUS photoproduction data, assessing the impact of our latest findings.

**KEYWORDS:** QCD, Jets.

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## 1. Introduction

One of the most commonly studied QCD observables is the flow of transverse energy ( $E_t$ ) into gaps between jets in various QCD hard processes. Since the  $E_t$  flow away from jets is infrared and collinear safe it is possible to make perturbative predictions for the same, which can be compared to experimental data for a given hard process. However since one is typically examining configurations where  $E_t$  is small compared to the hard scale  $Q$  of the process (e.g. jet transverse momenta in hadronic collisions) the perturbative predictions involve large logarithms in the ratio  $Q/E_t$ . Resummation of logarithmically enhanced terms of the form  $\alpha_s^n \ln^n(Q/E_t)$  has proved a challenge that is still to be fully met — complete calculations are available only in the large  $N_c$  limit [1–3]. Studies of the  $E_t$  flow have in fact directly led to developments in the theoretical understanding of QCD radiation and this process is still ongoing [4].

Another feature of the energy flow away from jets is its sensitivity to non-perturbative effects. Thus one may expect significant  $1/Q$  power corrections to energy flow distributions of a similar origin to those extensively studied for various jet-shape observables [5]. Moreover the  $E_t$  flow in hadronic collisions is a standard observable used to develop an understanding of the underlying event and to assess its role after accounting for perturbatively calculable QCD radiation [6, 7].

Given that  $E_t$  flow studies potentially offer so much valuable information on QCD over disparate scales, involving perturbative parameters such as the strong coupling  $\alpha_s$ , QCD evolution, coherence properties of QCD radiation and non-perturbative effects, it is not

surprising that they have been the subject of substantial theoretical effort over the past few years.

In this paper we wish to focus on the aspect of resummed predictions for the  $E_t$  flow into gaps between jets. Perhaps the most significant problem involved in making such predictions is the non-global nature of the observable [1, 2]. More precisely in order to resum the leading single logarithms involved, one has to address not just a veto on soft emissions coupled to the underlying primary hard parton antennae (known as the primary emission term), but additionally correlated emission or non-global contributions, where a clump of energy-ordered soft gluons coherently emit a still softer gluon into the gap region  $\Omega$ . For this latter contribution the highly non-trivial colour structure of the multi-gluon “hedgehog” configuration has proved at present too significant an obstacle to overcome. One thus has to resort to the large  $N_c$  limit to provide merely a leading-log estimate for the away-from-jet  $E_t$  flow. This situation can be contrasted with the case of event-shapes and Drell-Yan  $q_T$  resummations which have been pushed to next-to-leading and next-to-next-to-leading logarithmic accuracy respectively. The impact of finite  $N_c$  corrections in non-global observables is thus a factor in the theoretical uncertainty involved in the corresponding resummed predictions.

Given that the non-global component has a substantial quantitative impact over a significant range of  $E_t$  values for a given hard scale  $Q$  and that it is computable only in the large  $N_c$  approximation, it is clearly desirable to reduce the sensitivity of a given observable to non-global logarithms. An important observation in this regard was made in ref. [8]: if one employs the  $k_t$  clustering algorithm [9, 10] to define the final state such that the energy flow into a gap between jets is due to soft  $k_t$ -clustered mini-jets (rather than individual hadrons), the non-global logarithms are significantly reduced in magnitude.<sup>1</sup> This observation was exploited to study the case of  $E_t$  flow in dijet photoproduction where a result was provided for the primary emission component of the  $E_t$  distribution and the reduced non-global component was modeled [12].

However it has subsequently been found that  $k_t$  clustering also has a non-trivial impact on the primary emission component of the result [13]. This was not taken into account in refs. [8, 12] and also affects the ability to make resummed predictions for a host of other jet observables such as azimuthal correlation between jets  $\Delta\phi_{jj}$ . In fact the findings of ref. [13] are not just specific to the  $k_t$  algorithm but would also crop up in the case of jet observables defined using iterative cone algorithms.

In the present paper we wish to shed more light on the resummation of the primary or independent emission component of the result and its dependence on the clustering algorithm. While the leading  $\mathcal{O}(\alpha_s^2 \ln^2(Q/E_t))$  clustering dependent behaviour was computed analytically in ref. [13], the full resummed result for the primary emission component was computed only numerically in the case of a single hard emitting dipole ( $e^+e^- \rightarrow 2$  jets or DIS  $1 + 1$  jets). Here while sticking to a single hard dipole we shed more light on the structure of the primary emission term and analytically compute it to an accuracy that is sufficient for a wide range of phenomenological applications.

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<sup>1</sup>For recent progress on aspects of the  $k_t$  algorithm itself see ref. [11].

The analytical insight and calculations we provide here will also make the generalisation of the  $k_t$ -clustered primary emission result to the case of several hard emitters (dijets produced in photoproduction or hadron-hadron processes), involving a non-trivial colour flow, relatively straightforward.

The above resummation of the primary component of the answer assumes greater significance when we discuss our second observation: once an error is corrected in the numerical code used for the purposes of refs. [8, 12] the non-global component of the result is reduced even more compared to the earlier estimate. With a very small non-global component (which can be numerically computed in the large  $N_c$  limit) and a primary emission component that correctly treats the dependence on the jet algorithm, one is better placed to make more accurate resummed predictions than has been the case till now. This is true not just for the  $E_t$  flow but also as we mentioned for a variety of jet observables for which there are either no resummed predictions as yet, or only those employing jet algorithms not directly used in experimental studies [14].

This paper is organised as follows. In the following section we define the observable in question and revisit the issue of the dependence of the primary and non-global pieces on the jet clustering algorithm. Following this we demonstrate how the primary or independent emission piece can be computed at all orders in  $\alpha_s$ , accounting to sufficient accuracy for the effects of the clustering algorithm. We explicitly describe the case of three and four-gluon contributions to demonstrate the steps leading to our all-order results. Following this we re-examine the non-global component of the answer and find that this is significantly smaller than earlier calculations of the same [8]. We put our findings together to examine their impact on photoproduction data from the ZEUS collaboration [15] and lastly point to the conclusions one can draw and future extensions of our work.

## 2. Resummation of the primary emissions

Let us consider for simplicity the process  $e^+e^- \rightarrow 2$  jets. The calculations for processes involving a larger number of jets and more complex jet topologies can be done along similar lines.

We wish to examine the  $E_t$  flow in a region  $\Omega$  which we choose as a slice in rapidity<sup>2</sup> of width  $\Delta\eta$  which we can centre on  $\eta = 0$ . We then define the gap transverse energy as:

$$E_t = \sum_{i \in \Omega} E_{t,i}, \quad (2.1)$$

where the index  $i$  refers to soft jets obtained after  $k_t$  clustering of the final state. We shall concentrate on the integrated  $E_t$  cross-section which is defined as:

$$\Sigma(Q, Q_\Omega) = \frac{1}{\sigma} \int_0^{Q_\Omega} \frac{d\sigma}{dE_t} dE_t, \quad (2.2)$$

with  $\sigma$  the total cross-section for  $e^+e^- \rightarrow$  hadrons, with center-of-mass energy  $Q$ .

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<sup>2</sup>Since we are here dealing with back-to-back jets we can define the rapidity with respect to the jet axis or equivalently, for our purposes, the thrust axis.

The single-logarithmic result for the above, without  $k_t$  clustering (where the sum over  $i$  in eq. (2.1) refers to hadrons in the gap rather than jet clusters), was computed in ref. [2] and can be expressed as:

$$\Sigma(Q, Q_\Omega) = \Sigma_P(t) S(t), \quad t = \frac{1}{2\pi} \int_{Q_\Omega}^{Q/2} \frac{dk_t}{k_t} \alpha_s(k_t). \quad (2.3)$$

The above result contains a primary emission or ‘‘Sudakov’’ term<sup>3</sup>  $\Sigma_P(t)$  and a non-global term  $S(t)$ .

The primary emission piece is built up by considering only emissions attached to the primary hard partons namely those emitted from the hard initiating  $q\bar{q}$  dipole in our example, while the non-global term arises from coherent soft emission from a complex ensemble of soft emitters alongside the hard initiating dipole. More precisely we have:

$$\Sigma_P(t) = e^{-4C_F t \Delta\eta}, \quad (2.4)$$

which is the result of resumming uncanceled  $k_t$ -ordered virtual-emission contributions, in the gap region. The non-global component, as we stated before, is computed numerically in the large  $N_c$  limit.

Next we turn to the  $k_t$ -clustered case. The result stated in ref. [8] defines the primary or Sudakov piece to be unchanged by clustering since it appears to be the exponentiation of a single gluon emitted inside the gap. The remainder is recomputed numerically implementing clustering [8]. As already shown in ref. [13], this remainder comprises the genuine non-global contribution and corrections to primary emission due to the clustering. The corrections to the primary emission term first appear while considering two gluons emitted by the hard  $q\bar{q}$  dipole and persist at all orders. Below we provide a reminder of the two-gluon case discussed in ref. [13] and subsequently consider explicitly the three and four-gluon emission cases before writing down the result to all orders as a function of the radius parameter  $R$ .

## 2.1 Two-gluon emission

In order to examine the role of the  $k_t$  algorithm we point out that in our case ( $k_t$ -ordered soft limit) one can start the clustering procedure with the lowest transverse-energy parton or equivalently the softest parton. One examines the ‘‘distances’’ of this particle,  $i$ , from its neighbours, defined by  $d_{ij} = E_{t,i}^2 \left( (\Delta\eta_{ij})^2 + (\Delta\phi_{ij})^2 \right)$ , where  $E_{t,i}$  is the transverse energy of the softest parton. If the smallest of these distances is less than  $E_{t,i}^2 R^2$ , particle  $i$  is recombined or clustered into its nearest neighbour and the algorithm is iterated. On the other hand if all  $d_{ij}$  are greater than  $E_{t,i}^2 R^2$ ,  $i$  is counted as a jet and removed from the process of further clustering. The process continues until the entire final-state is made up of jets. Also in the limit of strong energy-ordering, which is sufficient to obtain the leading-logarithms we are concerned with here, the recombination of a softer particle with a harder one gives a jet that is aligned along the harder particle.

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<sup>3</sup>We use the term ‘‘Sudakov’’ in a loose sense since the primary emission result leads to an exponential that is analogous to a Sudakov form-factor.

The dependence of the primary emission term on the jet clustering algorithm starts naturally enough from the two-gluon level. While the Sudakov result  $\exp(-4C_F t \Delta\eta)$  comes about due to assuming real-virtual cancellations such that one is left with only virtual emissions with  $k_t \geq Q_\Omega$  in the gap region (for the integrated distribution),  $k_t$  clustering spoils this assumed cancellation.

Specifically let us take two real gluons  $k_1$  and  $k_2$  that are ordered in energy ( $\omega_1 \gg \omega_2$ ). We consider as in ref. [13] the region where the softer gluon  $k_2$  is in the gap whilst the harder  $k_1$  is outside. Additionally we take the case that the gluons are clustered by the jet algorithm which happens when  $(\Delta\eta)^2 + (\Delta\phi)^2 \leq R^2$  with  $\Delta\eta = \eta_2 - \eta_1$  and similarly for  $\Delta\phi$ , which condition we shall denote with the symbol  $\theta_{21}$ . Since  $k_2$  is clustered to  $k_1$  it gets pulled outside the gap, the recombined jet being essentially along  $k_1$ . Thus in this region the double real-emission term does not contribute to the gap energy *differential* distribution  $d\sigma/dE_t$ . Now let  $k_1$  be a virtual gluon. In this case it cannot cluster  $k_2$  out of the gap and we do get a contribution to the gap energy differential distribution. Thus a real-virtual cancellation which occurs for the unclustered case fails here and the mismatch for the integrated quantity  $\Sigma(t)$ , amounts to:

$$C_2^p = \frac{(-4C_F t)^2}{2!} \int_{k_1 \notin \Omega} d\eta_1 \frac{d\phi_1}{2\pi} \int_{k_2 \in \Omega} d\eta_2 \frac{d\phi_2}{2\pi} \theta_{21} = \frac{(-4C_F t)^2}{2!} \frac{2}{3\pi} R^3, \quad (2.5)$$

where we reported above the result computed for  $R \leq \Delta\eta$ , in ref. [13]. Here we introduced the primary emission term  $C_n^p$  that corrects the Sudakov result at  $\mathcal{O}(\alpha_s^n)$  due to the clustering requirement.

The fact that the result scales as the third power of the jet radius parameter is interesting in that by choosing a sufficiently small value of  $R$  one may hope to virtually eliminate this piece and thus the identification of the primary result with the Sudakov exponent would be at least numerically accurate. However the non-global term would then be significant which defeats the main use of clustering. If one chooses to minimise the non-global component by choosing e.g.  $R = 1$ , then one must examine the primary emission terms in higher orders in order to estimate their role. To this end we start by looking at the three and four-gluon cases below.

## 2.2 Three-gluon emission

Consider the emission of three energy-ordered gluons  $k_1, k_2$  and  $k_3$  with  $\omega_3 \ll \omega_2 \ll \omega_1$ , off the primary  $q\bar{q}$  dipole, and employing the inclusive  $k_t$  clustering algorithm [9, 10] as explained previously.

We consider all the various cases that arise when the gluons (which could be real or virtual) are in the gap region or outside. We also consider all the configurations in which the gluons are affected by the clustering algorithm. We then look for all contributions where a real-virtual mismatch appears due to clustering, that is not included in the exponential Sudakov term. The Sudakov itself is built up by integrating just virtual gluons in the gap, above the scale  $Q_\Omega$ . The corrections to this are summarised in table 1.

$\theta_{32}$	$\theta_{31}$	$\theta_{21}$	$k_3 \in \Omega$	$k_2 \in \Omega$	$k_1 \in \Omega$	$k_3, k_2 \in \Omega$	$k_3, k_1 \in \Omega$	$k_2, k_1 \in \Omega$
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	$W$	0
0	1	0	0	0	0	$W$	0	0
0	0	1	0	0	0	$W$	0	0
1	1	0	$W$	0	0	$W$	$W$	0
1	0	1	0	0	0	0	$W$	0
0	1	1	0	0	0	$W$	0	0
1	1	1	$W$	0	0	$W$	$W$	0

**Table 1:** Contributions of different configurations of particles to  $\Sigma_P(t)$  at  $\mathcal{O}(\alpha_s^3)$ . We define  $\theta_{ij} = \theta(R^2 - (\eta_i - \eta_j)^2 - (\phi_i - \phi_j)^2)$ , e.g.  $\theta_{13} = 1$  means  $(\eta_1 - \eta_3)^2 + (\phi_1 - \phi_3)^2 \leq R^2$ . We also define  $W = (-4C_F t)^3/3!$ , so the entries “ $W$ ” indicate a miscancellation which leads to a single-log correction to the Sudakov result, while the entries “0” indicate a complete real-virtual cancellation. We have discarded the case where all particles are in the gap since such configurations are already included in the exponential Sudakov result.

In order to obtain the various entries of the table one just looks at the angular configuration in question, draws all possible real and virtual contributions and looks for a mismatch between them generated by the action of clustering. We translate table 1 to:

$$\begin{aligned}
 C_3^p &= \frac{1}{3!} (-4C_F t)^3 \times \\
 &\times \left\{ \int_{k_1 \notin \Omega} d\eta_1 \frac{d\phi_1}{2\pi} \int_{k_2 \notin \Omega} d\eta_2 \frac{d\phi_2}{2\pi} \int_{k_3 \in \Omega} d\eta_3 \theta_{32} \theta_{31} + \right. \\
 &\quad + \int_{k_1 \notin \Omega} d\eta_1 \frac{d\phi_1}{2\pi} \int_{k_2 \in \Omega} d\eta_2 \frac{d\phi_2}{2\pi} \int_{k_3 \in \Omega} d\eta_3 [\theta_{31} + (1 - \theta_{31})(1 - \theta_{32})\theta_{21}] + \\
 &\quad \left. + \int_{k_1 \in \Omega} d\eta_1 \frac{d\phi_1}{2\pi} \int_{k_2 \notin \Omega} d\eta_2 \frac{d\phi_2}{2\pi} \int_{k_3 \in \Omega} d\eta_3 \theta_{32} \right\}, \tag{2.6}
 \end{aligned}$$

where we used the freedom to set  $\phi_3 = 0$ . We identify three equal contributions consisting of the integrals in which there is only one theta function constraining only two particles: the last integral over  $\theta_{32}$ , the integral over  $\theta_{31}$  and that over  $\theta_{21}$  in the third line. The set of configurations  $\theta_{32}$ ,  $\theta_{31}$  and  $\theta_{21}$  is just the set of constraints on all possible pairs of gluons, and in fact we can generalise the factor 3 to the case of any number  $n$  of gluons by  $n(n-1)/2$ , which will enable us to resum  $R^3$  terms. We shall return to this observation later. The integrals of the above type reduce essentially to the clustered two-gluon case as calculated in eq. (2.5), and the integral over the third “unconstrained” gluon is just  $\Delta\eta$ .

Explicitly we write eq. (2.6) as:

$$\begin{aligned}
 C_3^p &= \frac{1}{3!}(-4C_F t)^3 \times \\
 &\times \left\{ \int_{k_1 \notin \Omega} d\eta_1 \frac{d\phi_1}{2\pi} \int_{k_2 \notin \Omega} d\eta_2 \frac{d\phi_2}{2\pi} \int_{k_3 \in \Omega} d\eta_3 \theta_{32} \theta_{31} + \right. \\
 &\quad + \int_{k_1 \notin \Omega} d\eta_1 \frac{d\phi_1}{2\pi} \int_{k_2 \in \Omega} d\eta_2 \frac{d\phi_2}{2\pi} \int_{k_3 \in \Omega} d\eta_3 [\theta_{31}\theta_{32} - \theta_{31} - \theta_{32}] \theta_{21} + \\
 &\quad \left. + 3 \times \int_{k_1 \in \Omega} d\eta_1 \frac{d\phi_1}{2\pi} \int_{k_2 \notin \Omega} d\eta_2 \frac{d\phi_2}{2\pi} \int_{k_3 \in \Omega} d\eta_3 \theta_{32} \right\}. \tag{2.7}
 \end{aligned}$$

Computing the various integrals above (for simplicity we take  $R \leq \Delta\eta/2$ , which is sufficient for our phenomenological purposes) one obtains:

$$\begin{aligned}
 C_3^p &= \frac{1}{3!}(-4C_F t)^3 \times \\
 &\times \left\{ \left( \frac{\pi}{3} - \frac{32}{45} \right) \frac{R^5}{\pi^2} + f \frac{R^5}{\pi^2} - \left( \frac{\pi}{3} - \frac{32}{45} \right) \frac{R^5}{\pi^2} - \frac{32}{45} \frac{R^5}{\pi^2} + 3 \times \frac{2}{3\pi} \Delta\eta R^3 \right\}, \tag{2.8}
 \end{aligned}$$

with  $f \simeq 0.2807$  and we have written the results in the same order as the five integrals that arise from the various terms in eq. (2.7). Hence:

$$C_3^p = \frac{1}{3!}(-4C_F t)^3 \left\{ 3 \times \frac{2}{3\pi} \Delta\eta R^3 + f_2 R^5 \right\}, \tag{2.9}$$

where  $f_2 \simeq -0.04360$ . We note the appearance of an  $R^5$  term which, as we shall presently see, persists at higher orders. This term is related to a clustering constraint on *three* gluons at a time via the product of step functions  $\theta_{32} \theta_{21} (\theta_{31} - 1)$  with  $k_2, k_3 \in \Omega$  and  $k_1 \notin \Omega$ .

Next we look at the emission of four soft, real or virtual energy-ordered gluons. This will help us move to a generalisation with any number of gluons.

### 2.3 Four-gluon case and beyond

Now we take the case of four-gluon emission and identify the patterns that appear at all orders. A table corresponding to table 1 is too lengthy to present here. The result can



however be expressed in an equation similar to that for the three-gluon case. We have:

$$\begin{aligned}
 C_4^p &= \frac{1}{4!}(-4C_{Ft})^4 \times \\
 &\times \left\{ \int_{1 \text{ in}} \int_{2 \text{ in}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{43} + \right. \\
 &\quad + \int_{1 \text{ in}} \int_{2 \text{ out}} \int_{3 \text{ in}} \int_{4 \text{ in}} [\theta_{42} + \theta_{32}(1 - \theta_{43})(1 - \theta_{42})] + \\
 &\quad + \int_{1 \text{ out}} \int_{2 \text{ in}} \int_{3 \text{ in}} \int_{4 \text{ in}} \{ \theta_{41} + \theta_{-41} [\theta_{31}\theta_{-43} + \theta_{43}\theta_{21}\theta_{-42} + \theta_{21}\theta_{-42}\theta_{-43}\theta_{-31}\theta_{-32}] \} + \\
 &\quad + \int_{1 \text{ in}} \int_{2 \text{ out}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{42}\theta_{43} + \\
 &\quad + \int_{1 \text{ out}} \int_{2 \text{ in}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{43} [\theta_{41} + \theta_{-41}\theta_{-42}\theta_{21}] + \\
 &\quad + \int_{1 \text{ out}} \int_{2 \text{ out}} \int_{3 \text{ in}} \int_{4 \text{ in}} \theta_{41}\theta_{42} + \theta_{41}\theta_{-42}\theta_{-43}\theta_{32} + \theta_{-41}\theta_{-43}\theta_{31} [\theta_{42} + \theta_{-42}\theta_{32}] + \\
 &\quad \left. + \int_{1 \text{ out}} \int_{2 \text{ out}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{41}\theta_{42}\theta_{43} \right\}, \tag{2.10}
 \end{aligned}$$

where  $\theta_{-ij} = 1 - \theta_{ij}$  and “in” or “out” pertains to whether the gluon is inside the gap region or out. For brevity we did not write the differential phase-space factor for each gluon which is as always  $d\eta d\phi/(2\pi)$ . We identify six  $R^3$  terms exactly of the same kind as computed before and similarly four  $R^5$  terms. Explicitly we have:

$$\begin{aligned}
 C_4^p &= \frac{1}{4!}(-4C_{Ft})^4 \times \\
 &\times \left\{ 6 \times \int_{1 \text{ in}} \int_{2 \text{ in}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{43} + \right. \\
 &\quad + 4 \times \left( \int_{1 \text{ in}} \int_{2 \text{ out}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{42}\theta_{43} + \int_{1 \text{ in}} \int_{2 \text{ out}} \int_{3 \text{ in}} \int_{4 \text{ in}} \theta_{32} [\theta_{43}\theta_{42} - \theta_{43} - \theta_{42}] \right) + \\
 &\quad + 3 \times \int_{1 \text{ out}} \int_{2 \text{ in}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{21}\theta_{43} [1 - \theta_{41} - \theta_{42} + \theta_{41}\theta_{42}] + \\
 &\quad + \int_{1 \text{ out}} \int_{2 \text{ in}} \int_{3 \text{ in}} \int_{4 \text{ in}} \theta_{21} [\theta_{42}\theta_{43} - \theta_{42} - \theta_{43} - \theta_{41}\theta_{-42}\theta_{-43}] [\theta_{31}\theta_{32} - \theta_{31} - \theta_{32}] + \\
 &\quad + \int_{1 \text{ out}} \int_{2 \text{ out}} \int_{3 \text{ in}} \int_{4 \text{ in}} \theta_{32}\theta_{31} [\theta_{41}(1 - \theta_{43})(\theta_{42} - 2) - \theta_{43}] + \\
 &\quad \left. + \int_{1 \text{ out}} \int_{2 \text{ out}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{41}\theta_{42}\theta_{43} \right\}. \tag{2.11}
 \end{aligned}$$

We discuss below each set of integrals, generalise the result to the case of  $n$  emitted gluons and then resum all orders.

- The integral:

$$\frac{1}{4!}(-4C_{Ft})^4 6 \times \int_{1 \text{ in}} \int_{2 \text{ in}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{43}. \tag{2.12}$$

The integrals over particles 1 and 2 give  $(\Delta\eta)^2$ . The remaining integrals reduce to the result computed for the two-gluon case, i.e. the  $R^3$  term, multiplied by a factor

of 6 accounting for the number of pairs of gluons  $n(n-1)/2$ , for  $n = 4$ . Explicitly we have for this term:

$$\frac{1}{4!}(-4C_F t)^4 \frac{4 \times 3}{2} \Delta\eta^{4-2} \frac{2}{3\pi} R^3. \quad (2.13)$$

For  $n$  emitted gluons the  $R^3$  term, which is always related to the clustering of two gluons, is given by:

$$\frac{1}{n!} \frac{n(n-1)}{2} (-4C_F t \Delta\eta)^n \Delta\eta^{-2} \frac{2}{3\pi} R^3, \quad n \geq 2. \quad (2.14)$$

Hence to all orders one can sum the above to obtain:

$$e^{-4C_F t \Delta\eta} \frac{(-4C_F t)^2}{2} \frac{2}{3\pi} R^3. \quad (2.15)$$

- The integrals:

$$\frac{1}{4!}(-4C_F t)^4 4 \times \left( \int_{1 \text{ in}} \int_{2 \text{ out}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{42} \theta_{43} + \int_{1 \text{ in}} \int_{2 \text{ out}} \int_{3 \text{ in}} \int_{4 \text{ in}} \theta_{32} [\theta_{43} \theta_{42} - \theta_{43} - \theta_{42}] \right). \quad (2.16)$$

The integral over particle 1 gives  $\Delta\eta$ , while the rest of the integrals reduce to the ones calculated earlier which gave the  $R^5$  result, accompanied with a factor of 4 standing for the number of triplet combinations formed by four gluons. For  $n$  emitted gluons this factor is  $n(n-1)(n-2)/3!$ . Explicitly we have for this case:

$$\frac{1}{4!}(-4C_F t)^4 \frac{4 \times 3 \times 2}{6} \Delta\eta^{4-3} f_2 R^5. \quad (2.17)$$

At the  $n^{\text{th}}$  order we obtain:

$$\frac{1}{n!}(-4C_F t \Delta\eta)^n \frac{n(n-1)(n-2)}{6} \Delta\eta^{-3} f_2 R^5, \quad n \geq 3. \quad (2.18)$$

Summing all orders we get:

$$e^{-4C_F t \Delta\eta} \frac{(-4C_F t)^3}{6} f_2 R^5. \quad (2.19)$$

- The integral:

$$\frac{1}{4!}(-4C_F t)^4 3 \times \int_{1 \text{ out}} \int_{2 \text{ in}} \int_{3 \text{ out}} \int_{4 \text{ in}} \theta_{21} \theta_{43}. \quad (2.20)$$

This integral can be factored into two separate integrals involving the constraint on  $k_1$  and  $k_2$  and over  $k_3$  and  $k_4$  respectively. Each of these reduces to the  $R^3$  result obtained in the two-gluon case. Thus we get:

$$\frac{1}{4!}(-4C_F t)^4 3 \times \left( \frac{2}{3\pi} \right)^2 R^6. \quad (2.21)$$

At  $n^{\text{th}}$  order this becomes:

$$\frac{1}{n!} \frac{n(n-1)(n-2)(n-3)}{8} (-4C_{Ft}\Delta\eta)^n \Delta\eta^{-4} \left(\frac{2}{3\pi}\right)^2 R^6, \quad n \geq 4, \quad (2.22)$$

which can be resummed to:

$$e^{-4C_{Ft}\Delta\eta} \frac{(-4C_{Ft})^4}{8} \left(\frac{2}{3\pi}\right)^2 R^6. \quad (2.23)$$

The factor 3 (and generally  $n(n-1)(n-2)(n-3)/8$ ) is the number of configurations formed by four (and generally  $n$ ) gluons such that we have two pairs of gluons each is formed by an out-of-gap gluon connected to a softer in-gap one.

- The remaining integrals give at most an  $\mathcal{O}(R^7)$  term because they constrain all the four gluons at once. In fact for gap sizes  $\Delta\eta \geq 3R$ , these integrals go purely as  $R^7$  with no dependence on  $\Delta\eta$ . Since here however we wish to use the condition  $\Delta\eta \geq 2R$ , which allows us to make use of the whole range of HERA data, these integrals do not depend purely on  $R$  but are a function of  $R$  and  $\Delta\eta$  which have an upper bound of order  $R^7$ . This can be seen by noting that there are three azimuthal integrations that each produce a function which has a maximum value proportional to  $R$ , so the result of integrating over all azimuthal variables is a factor that is bounded from above by  $R^3$ . Similarly there are four rapidity integrations with a clustering constraint on all four gluons implying that they can produce an  $R^4$  term at most. In general the result at  $n^{\text{th}}$  order of constraining  $n$  gluons at once, is bounded from above by a factor of order  $R^{2n-1}$ .

We can write the result for all these as  $(-4C_{Ft})^4/4! y(R, \Delta\eta)$ , and resum such terms to all orders (in the same manner as before) to:

$$e^{-4C_{Ft}\Delta\eta} \frac{(-4C_{Ft})^4}{4!} y(R, \Delta\eta), \quad (2.24)$$

where  $y(R, \Delta\eta)$  is at most  $\mathcal{O}(R^7)$ . We do not calculate these terms (though it is possible to do so) since the accuracy we achieve by retaining the  $R^3$ ,  $R^5$  and  $R^6$  terms, we have already computed, is sufficient as we shall show.

The five-gluon case is too lengthy to analyse here. The same patterns as pointed out above persist here but new terms that are at most  $\mathcal{O}(R^9)$  appear when all five gluons are constrained. There is also an  $R^8$  term, coming from the combination of  $R^3$  and  $R^5$  terms in the same manner that the  $R^6$  term arose as a combination of two  $R^3$  terms.

### 3. All-orders result

From the above observations we can assemble an all-orders result to  $R^6$  accuracy, where we shall consider  $R$  to be at most equal to unity. The final result for primary emissions

alone and including the usual Sudakov logarithms (for  $\Delta\eta \geq 2R$ ) is:

$$\Sigma_P(t) = e^{-4C_F t \Delta\eta} \times \left( 1 + (-4C_F t)^2 \frac{1}{3\pi} R^3 + (-4C_F t)^3 \frac{f_2}{6} R^5 + (-4C_F t)^4 \frac{1}{18\pi^2} R^6 + \frac{(-4C_F t)^4}{4!} \mathcal{O}(R^7) \right). \quad (3.1)$$

Formally one may wish to extend this accuracy by computing a few more terms such as those integrals that directly give or are bounded by an  $R^7$  behaviour and this is possible though cumbersome. It should also be unnecessary from a practical viewpoint, especially keeping in mind that  $R = 0.7$  is a preferable value to  $R = 1$ , in the important case of hadron collisions<sup>4</sup> and the fact that even at  $R = 1$  the  $R^3$  term significantly dominates the result over the range of  $t$  values of phenomenological interest, as we shall see below.

We further note that if one keeps track of all the terms that come about as a combination of  $R^3$  and  $R^5$  terms in all possible ways at all orders, one ends up with the following form for eq. (3.1):

$$\Sigma_P(t) = e^{-4C_F t \Delta\eta} \exp \left( \frac{(-4C_F t)^2}{2!} \frac{2}{3\pi} R^3 + \frac{(-4C_F t)^3}{3!} f_2 R^5 + \frac{(-4C_F t)^4}{4!} \mathcal{O}(R^7) \right), \quad (3.2)$$

the expansion of which agrees with eq. (3.1). In the above by  $\mathcal{O}(R^7)$  we mean terms that, while they may depend on  $\Delta\eta$ , are at most as significant as an  $R^7$  term. We also mention that in the formal limit  $\Delta\eta \rightarrow \infty$ , there is no dependence of the clustering terms on  $\Delta\eta$  and they are a pure power series in  $R$ . The limit of an infinite gap appears in calculations where the region considered includes one of the hard emitting partons. An example of such cases (which have a leading double-logarithmic behaviour) is once again the quantity  $\Delta\phi_{jj}$  between jets in e.g. DIS or hadron collisions.

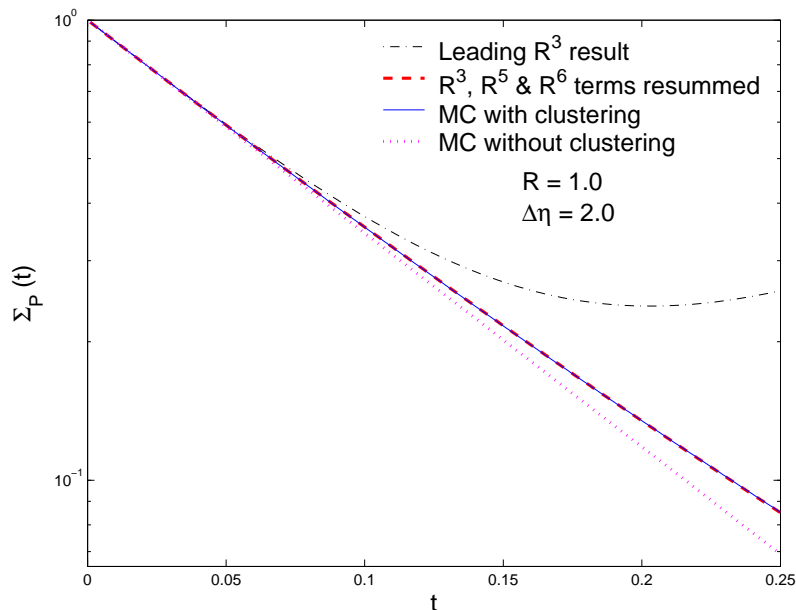
Figure 1 represents a comparison between the leading  $R^3$  result (i.e. the pure fixed-order result of ref. [13] combined with the resummed Sudakov exponent), the resummed  $R^3$ ,  $R^5$  and  $R^6$  result (eq. (3.1)) and a numerical Monte Carlo estimate with and without clustering. The Monte Carlo program in question is essentially that described in ref. [1], with the modification of  $k_t$  clustering where we computed just emissions off the primary dipole “switching off” the non-global correlated emission.

We note that the resummed analytical form (3.1) is in excellent agreement with the numerical result which contains the full  $R$  dependence. We have tested this agreement with a range of values of  $R$ . We take this agreement as indicating that uncomputed  $R^7$  and higher terms can safely be ignored even at  $R = 1$  and even more so at fractional values of  $R$ , e.g.  $R = 0.7$ . To provide an idea about the relative role of terms at different powers of  $R$  in eq. (3.1) we note that for  $R = 1$  and  $t = 0.25$  the resummed  $R^3$  term increases the Sudakov result  $\exp(-4C_F t \Delta\eta)$  by 19%, the  $R^5$  term represents a further increase of 1.5% to the result after inclusion of the resummed  $R^3$  term and the  $R^6$  term has a similar effect on the result obtained after including up to  $R^5$  terms.

Next we comment on the size of the non-global component at different values of  $R$ .

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<sup>4</sup>This is because the underlying event will contaminate jets less if one chooses a smaller  $R$ .



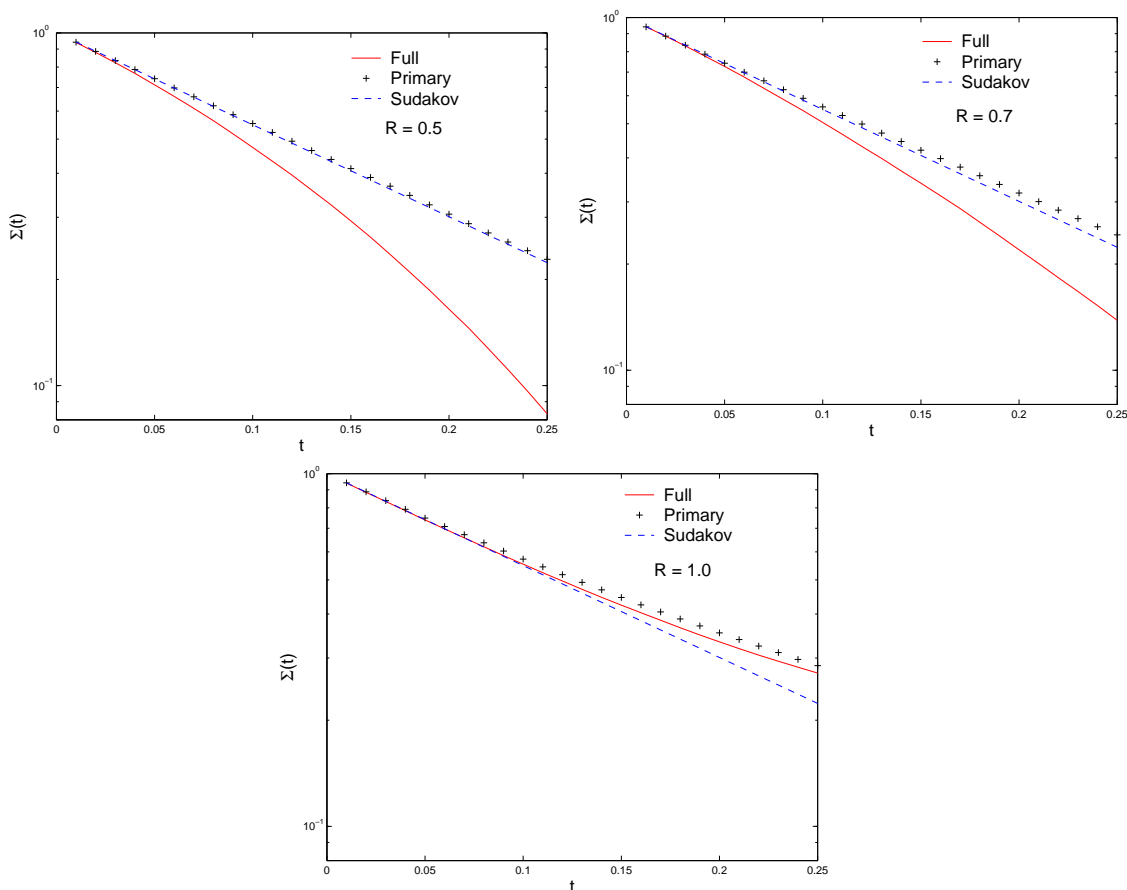
**Figure 1:** Comparison of the analytical results to a numerical Monte Carlo estimate.

#### 4. Revisiting the non-global contribution

We have seen above how the primary emission piece is dependent on the jet clustering algorithm. It was already noted in ref. [8] that the non-global contribution is significantly reduced by clustering. Here we wish to point out that after correction of an oversight in the code used there, the non-global component is even more significantly reduced than previously stated in the literature. Indeed for  $R = 1$  and the illustrative value of  $t = 0.15$ , which corresponds to gap energy  $Q_\Omega = 1$  GeV for a hard scale  $Q = 100$  GeV, the non-global logarithms are merely a 5% effect as opposed to the 20% reported previously [8] and the over 65% effect in the unclustered case.

In figure 2 we plot the curves for the primary and full results (in the large  $N_c$  limit) for the integrated quantity  $\Sigma(t)$  as a function of  $t$  defined earlier. We note that for  $R = 0.5$  the primary result is essentially identical to the Sudakov result. The non-global contribution (which is the ratio of the full and primary curves) is however still quite significant. Neglecting it leads to an overestimate of 40% for  $t = 0.15$ . Increasing the jet radius in a bid to lower the non-global component we note that for  $R = 0.7$  the impact of the non-global component is now just over 20% while the difference between the full primary result and the Sudakov result is small (less than 5%). The situation for  $R = 1$  is a bit different. Here it is the non-global logarithms that are only a 5% effect (compared to the 20% claimed earlier [8]) while the full primary result is bigger than the Sudakov term by around 11%.

The value  $R = 1$  is in fact the one used in the HERA analyses of gaps-between-jets in photoproduction. It is now clear that such analyses will have a very small non-global component and a moderate effect on primary emissions due to clustering. In order to completely account for the primary emission case for dijet photoproduction one would

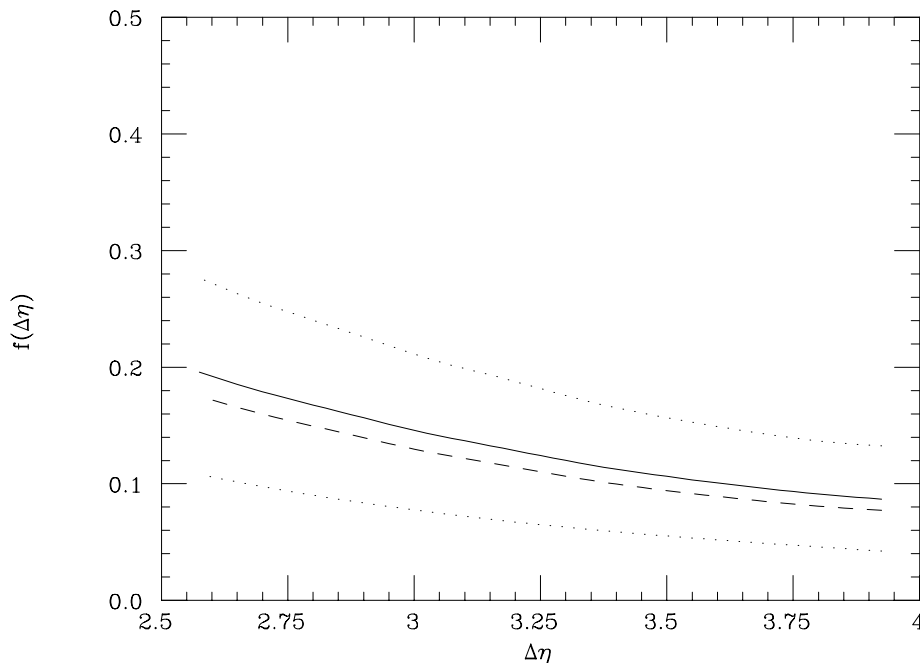


**Figure 2:** Comparison of the Sudakov result, the correct primary result and the full result including non-global logarithms, for different values of  $R$  and with  $\Delta\eta = 1$ . All quantities are shown in the large  $N_c$  limit for ease of comparison.

need to generalise the calculations presented here for a single  $q\bar{q}$  dipole to the case of several hard emitting dipoles. An exactly similar calculation would be needed for the case of hadron-hadron collisions and this is work in progress. It is straightforward however to at least estimate the effect of our findings on the photoproduction case and we deal with this in the following section.

## 5. Gaps between jets at HERA – the ZEUS analysis

We can test the perturbative framework presented in this paper with energy flow measurements in the photoproduction of dijets. These energy flow observables are defined with two hard jets in the central detector region separated by a gap in pseudorapidity. A gap event is defined when the sum of the hadronic transverse energy in the gap is less than a cut-off, and the gap fraction is defined as the ratio of the gap cross-section to the total inclusive cross-section. The energy flow observables measured by ZEUS [15] and H1 [16] use the  $k_t$  clustering definition of the hadronic final state, and the transverse energy in the gap is given by the sum of the mini-jet contributions. In this paper we focus on the



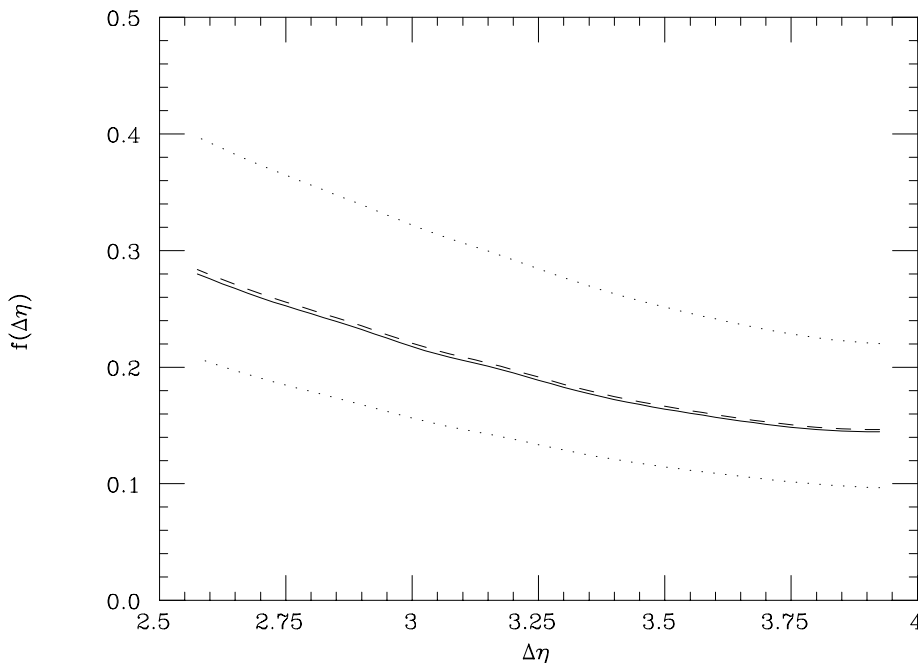
**Figure 3:** The gap fraction for the ZEUS analysis with a  $k_t$ -defined final state ( $R = 1.0$  and  $Q_\Omega = 0.5$  GeV). The solid line shows the effect of resummed primary emission, the primary emission clustering correction factor and the non-global suppression factor. The overall theoretical uncertainty in all three contributions is shown by the dotted lines. The dashed line indicates the gap fraction obtained by only including primary resummed emission without accounting for clustering.

ZEUS measurements and provide revised theoretical estimates for them. These revisions lead to changes that are minor in the context of the overall theoretical uncertainty but should become more significant once the matching to fixed higher-orders is carried out and an estimate of the next-to-leading logarithms is obtained. The H1 data was considered in ref. [12], where the theoretical analysis consisted of only the resummed primary emission contribution without taking account of the effect of  $k_t$  clustering.

The ZEUS data was obtained by colliding 27.5 GeV positrons with 820 GeV protons, with a total integrated luminosity of  $38.6 \pm 1.6$  pb $^{-1}$  in the 1996-1997 HERA running period. The full details of the ZEUS analysis can be found in ref. [15], but the cuts relevant to the calculations in this paper are:

$$\begin{aligned}
 &0.2 < y < 0.75, \\
 &Q^2 < 1 \text{ GeV}^2, \\
 &6, 5 \text{ GeV} < E_T(1, 2), \\
 &|\eta(1, 2)| < 2.4, \\
 &|0.5(\eta_1 + \eta_2)| < 0.75, \\
 &2.5 < \Delta\eta < 4,
 \end{aligned}$$

where  $y$  is the inelasticity,  $Q^2$  is the virtuality of the photon,  $E_T(1, 2)$  are the transverse



**Figure 4:** The gap fraction for the ZEUS analysis with a  $k_t$ -defined final state ( $R = 1.0$  and  $Q_\Omega = 1.0$  GeV). The solid line shows the effect of resummed primary emission, the primary emission clustering correction factor and the non-global suppression factor. The overall theoretical uncertainty in all three contributions is shown by the dotted lines. The dashed line indicates the gap fraction obtained by only including primary resummed emission without accounting for clustering.

energies of the two hard jets,  $\eta(1, 2)$  are the pseudorapidities of the two hardest jets and  $\Delta\eta$  is the jet rapidity difference. The further requirement for the gap sample is  $E_{t, \text{gap}} < Q_\Omega = 0.5, 1, 1.5, 2$  GeV, and the clustering parameter  $R$  is always taken to be unity.

The theoretical prediction for the gap fraction is composed of the primary piece, with corrections due to clustering, and the non-global piece. We shall now describe each in turn.

The resummed primary contribution ignoring the clustering corrections, is obtained from the factorisation methods of Sterman et al [14] and is described in ref. [12]. The four-jet case of photoproduction requires a matrix formalism, and the exponents of the Sudakov factors in the gap cross-section are anomalous dimension matrices over the basis of possible colour flows of the hard sub-process. The emission of soft gluons cause mixing of the colour basis. Consideration of the eigenvectors and eigenvalues of the anomalous dimension matrices, together with sub-process-dependent hard and soft matrices, allows the resummed four-jet primary emission differential cross-section to be written as [12]:

$$\frac{d\sigma}{d\eta} = \sum_{L,I} H_{IL} S_{LI} \exp \left\{ (\lambda_L^*(\eta, \Omega) + \lambda_I(\eta, \Omega)) \int_{p_t}^{Q_\Omega} \frac{d\mu}{\mu} \alpha_s(\mu) \right\}, \quad (5.1)$$

where  $H$  and  $S$  denote the hard and soft matrices (defined in the colour basis in which the anomalous dimension matrices are diagonal),  $\lambda$  denotes the eigenvalues of the anomalous dimension matrices,  $\eta = \Delta\eta/2$  and  $p_t$  is the hard scale of the process. This was computed



in ref. [12] for the case of photoproduction and energy flow observables measured by H1. In this paper we have recomputed this differential gap cross-section for the observable defined by the ZEUS collaboration. The uncertainty in the renormalisation scale is quantified by varying the hard scale in the resummation by a factor of 2 (upper bound) and 0.5 (lower bound).

We now need to account for the effect of clustering on eq. (5.1). Since we do not have as yet the full results for the four-jet case of photoproduction we simply estimate the full correction as the square of the correction arising in the two-jet case dealt with here, using the appropriate colour factors for each hard sub-process. This was also the method used to approximate the non-global contribution for the four-jet case in ref. [12]. While we emphasise that this is only a rough way of examining the impact of the clustering dependent terms computed here, given the size of the effects we are dealing with, it is clear that no significant differences ought to emerge if one were to properly compute the various dipole sub-processes we need to account for. We also include the revised and virtually negligible non-global component in an identical fashion to arrive at the best current theoretical estimates.

The results for the ZEUS gap-fraction with a  $k_t$ -defined final state are shown in figures 3 and 4. We consider here two different values for the gap energy  $Q_\Omega$ . For the value of  $Q_\Omega = 0.5$  GeV one notes that the full prediction accounting approximately for all additional sources of single-logarithmic enhancements, is somewhat higher than the pure ‘‘Sudakov’’ type prediction. This is due to the extra primary terms we compute here, non-global corrections being negligible. For a larger value of  $Q_\Omega = 1.0$  GeV the difference between the clustered and unclustered primary results is negligible. We also note the large theoretical uncertainty on the prediction as represented by the renormalisation scale dependence. This is to be expected in light of the fact that the predictions here are not matched to fixed-order and account only for the leading logarithms. Improvements along both these directions should be possible in the immediate future after which the role of the various effects we highlighted here should be revisited.

## 6. Conclusions

In the present paper we have shed further light on resummations of  $k_t$ -clustered final states. We have shown that both the primary and non-global components of the resummed result are affected by clustering and dealt with the resummation of each in turn. For the non-global component we find that the results after applying clustering are different from those presented earlier [8]. The new results we present here indicate an even smaller non-global component than previously believed.

We have also shown how the primary emission clustering effects can be resummed to all orders as an expansion in the clustering parameter  $R$  and computed a few terms of the series. The analytical results we have provided here for a single emitting dipole should be generalisable to the case of several hard dipoles (multi-jet processes). This should then enable one to write a correct resummed result for primary emissions to a high accuracy and deal with the reduced non-global component in the large  $N_c$  limit. Such progress is

relevant not just to energy-flow studies but to any jet observable of a non-global nature, requiring resummation. An example is the azimuthal angle  $\Delta\phi_{jj}$  between jets, mentioned previously. The work we have carried out should enable next-to-leading log calculations of such jet observables to sufficient accuracy to enable phenomenological studies of the same.

Lastly we have also mentioned the impact of the new findings on the ZEUS gaps-between-jets analysis. Since the non-global effects are very small for  $R = 1$  the main new effect is the additional clustering dependent primary terms we computed here. Approximating the effect of these terms for the case of photoproduction, somewhat changes the theoretical predictions but this change is insignificant given the large theoretical uncertainty that arises due to missing higher orders and unaccounted for next-to-leading logarithms. We consider both these areas as avenues for further work and hope that more stringent comparisons can thus be made in the very near future.

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